Newton's method

Newton's method lets us approximate the solution of a function, which is the point where the function crosses the x-axis. Keep the following in mind when you use Newton's method:

- The function must be in the form $f(x) = 0$.
- The more approximations we take, the closer we'll get to the actual solution.
- For each approximation, we have to use our answer from the previous approximation.

The Newton's method formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where x_n is our starting approximation and x_{n+1} is our next approximation. Once x_n and x_{n+1} are exactly equal or equal to a specified number of decimal places, we can stop taking approximations.

Example

Use Newton's method to find an approximation of the root of the function to four decimal places.

$$x^2 = x$$

when $x_0 = -1$

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First we verify that our equation is in the form $f(x) = 0$.

$$x^2 - x = 0$$

Next we take the derivative of our function.

$$2x - 1 = 0$$

Since we were given x_0, we can set $x_n = x_0$ and $x_{n+1} = x_1$, and we get

$$x_1 = x_0 - \frac{(x_0)^2 - x_0}{2x_0 - 1}$$

We were given $x_0 = -1$, so we'll plug that in and get

$$x_1 = -1 - \frac{(-1)^2 - (-1)}{2(-1) - 1}$$

$$x_1 = -1 - \frac{1 + 1}{-2 - 1}$$

$$x_1 = -1 - \frac{2}{-3}$$

$$x_1 = -1 + \frac{2}{3}$$

$$x_1 = -\frac{1}{3} \approx -0.3333$$

We'll use the value we just found, $x_1 = -1/3$ in our next approximation, where we'll solve for x_2.

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$$x_2 = x_1 - \frac{(x_1)^2 - x_1}{2x_1 - 1}$$

$$x_2 = -\frac{1}{3} - \frac{\left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right)}{2\left(-\frac{1}{3}\right) - 1}$$

$$x_2 = -\frac{1}{3} - \frac{\frac{1}{9} + \frac{1}{3}}{-\frac{2}{3} - 1}$$

$$x_2 = -\frac{1}{3} - \frac{\frac{1}{9} + \frac{3}{9}}{-\frac{2}{3} - \frac{3}{3}}$$

$$x_2 = -\frac{1}{3} - \frac{\frac{4}{9}}{-\frac{5}{3}}$$

$$x_2 = -\frac{1}{3} + \frac{\frac{4}{9}}{\frac{5}{3}}$$

$$x_2 = -\frac{1}{3} + \frac{4}{9}\left(\frac{3}{5}\right)$$

$$x_2 = -\frac{1}{3} + \frac{4}{3}\left(\frac{1}{5}\right)$$

$$x_2 = -\frac{1}{3} + \frac{4}{15}$$

$$x_2 = -\frac{5}{15} + \frac{4}{15}$$

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$$x_2 = -\frac{1}{15} \approx -0.0667$$

We'll use $x_2 = -1/15$ to solve for x_3.

$$x_3 = x_2 - \frac{(x_2)^2 - x_2}{2x_2 - 1}$$

$$x_3 = -\frac{1}{15} - \frac{\left(-\frac{1}{15}\right)^2 - \left(-\frac{1}{15}\right)}{2\left(-\frac{1}{15}\right) - 1}$$

$$x_3 = -\frac{1}{15} - \frac{\frac{1}{225} + \frac{1}{15}}{-\frac{2}{15} - 1}$$

$$x_3 = -\frac{1}{15} - \frac{\frac{1}{225} + \frac{15}{225}}{-\frac{2}{15} - \frac{15}{15}}$$

$$x_3 = -\frac{1}{15} - \frac{\frac{16}{225}}{-\frac{17}{15}}$$

$$x_3 = -\frac{1}{15} + \frac{\frac{16}{225}}{\frac{17}{15}}$$

$$x_3 = -\frac{1}{15} + \frac{16}{225}\left(\frac{15}{17}\right)$$

$$x_3 = -\frac{1}{15} + \frac{16}{15}\left(\frac{1}{17}\right)$$

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$$x_3 = -\frac{1}{15} + \frac{16}{255}$$

$$x_3 = -\frac{17}{255} + \frac{16}{255}$$

$$x_3 = -\frac{1}{255} \approx -0.0039$$

We'll use $x_3 = -1/255$ to solve for x_4.

$$x_4 = x_3 - \frac{(x_3)^2 - x_3}{2x_3 - 1}$$

$$x_4 = -\frac{1}{255} - \frac{\left(-\frac{1}{255}\right)^2 - \left(-\frac{1}{255}\right)}{2\left(-\frac{1}{255}\right) - 1}$$

$$x_4 = -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{1}{255}}{-\frac{2}{255} - 1}$$

$$x_4 = -\frac{1}{255} - \frac{\frac{1}{255^2} + \frac{255}{255^2}}{-\frac{2}{255} - \frac{255}{255}}$$

$$x_4 = -\frac{1}{255} - \frac{\frac{256}{255^2}}{-\frac{257}{255}}$$

$$x_4 = -\frac{1}{255} + \frac{\frac{256}{255^2}}{\frac{257}{255}}$$

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$$x_4 = -\frac{1}{255} + \frac{256}{255^2}\left(\frac{255}{257}\right)$$

$$x_4 = -\frac{1}{255} + \frac{256}{255}\left(\frac{1}{257}\right)$$

$$x_4 = -\frac{1}{255} + \frac{256}{255 \cdot 257}$$

$$x_4 = -\frac{257}{255 \cdot 257} + \frac{256}{255 \cdot 257}$$

$$x_4 = -\frac{1}{255 \cdot 257}$$

$$x_4 = -\frac{1}{65{,}535} \approx -0.0000$$

The solution is only getting smaller, which means that the next approximation will also be all zeros to the first four decimal places. Therefore, the approximation of the root of the function to four decimal places is 0.

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